H-3

Hyperbolas

What you'll learn about

- Geometry of a Hyperbola
- Translations of Hyperbolas
- Eccentricity and Orbits
- Reflective Property of a Hyperbola
- Long-Range Navigation

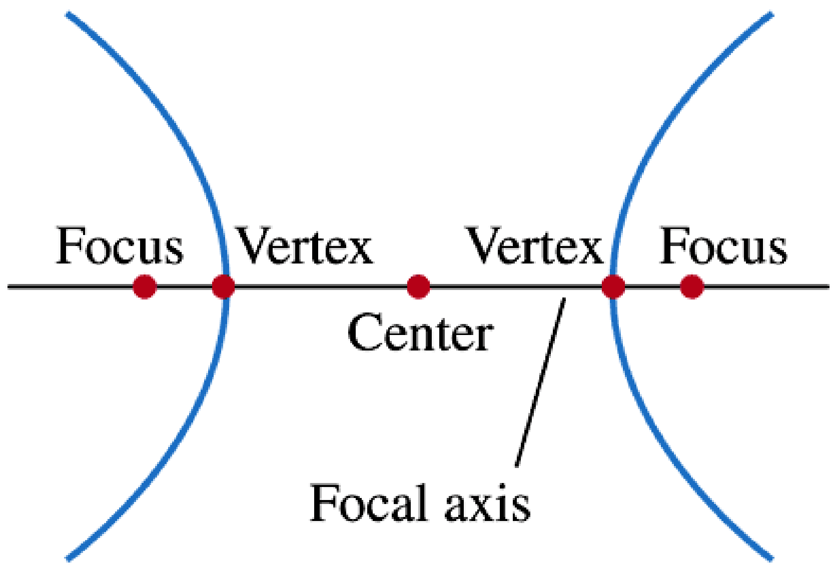
… and why

The hyperbola is the least known conic section, yet it is used astronomy, optics, and navigation.

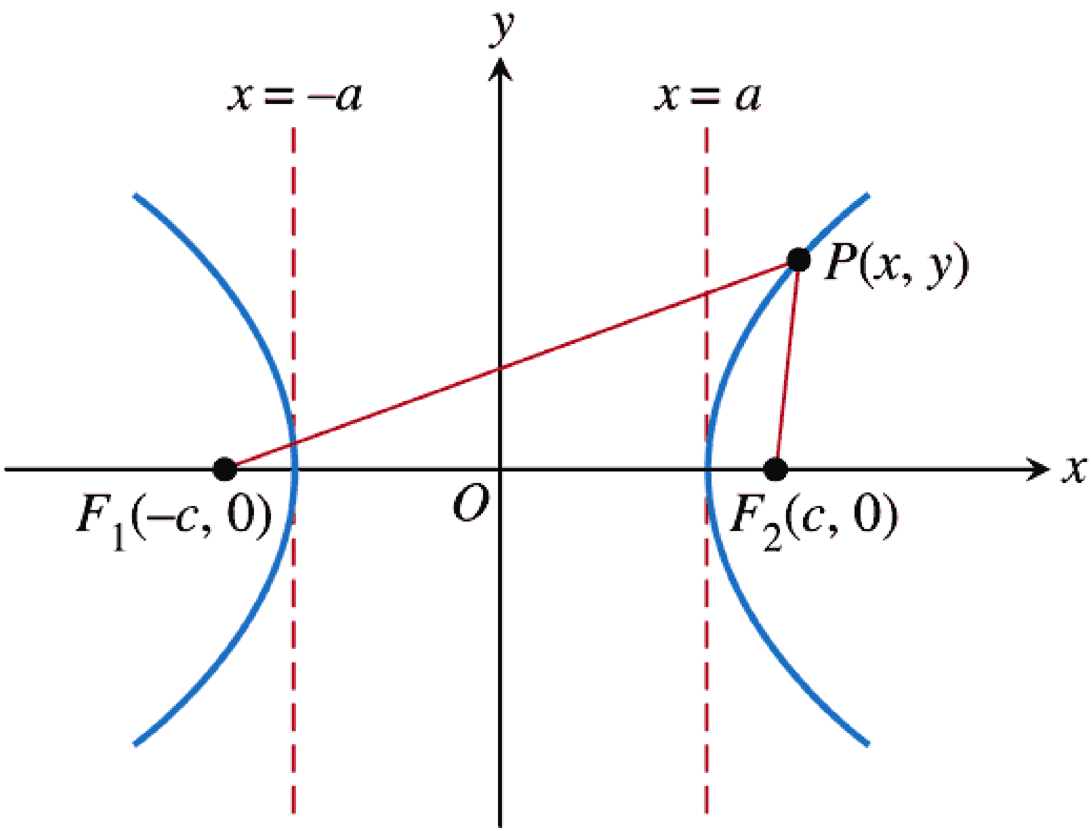
Hyperbola

A **hyperbola** is the set of all points in a plane whose distances from two fixed points in the plane have a constant difference. The fixed points are the **foci** of the hyperbola. The line through the foci is the **focal axis**. The point on the focal axis midway between the foci is the **center**. The points where the hyperbola intersects its focal axis are the **vertices** of the hyperbola.

Hyperbola



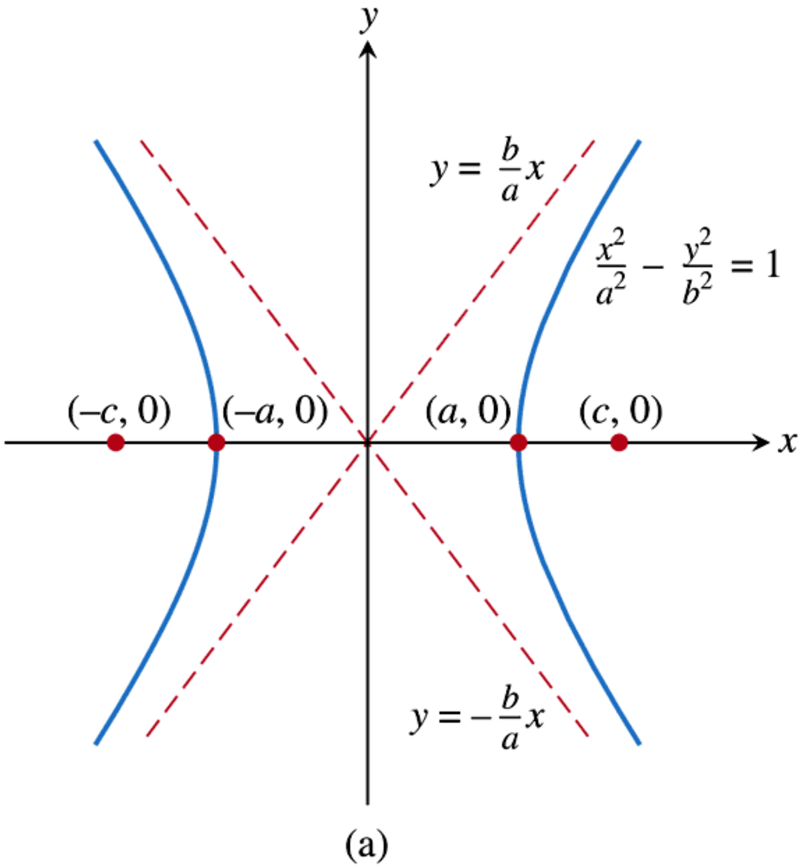
Hyperbola

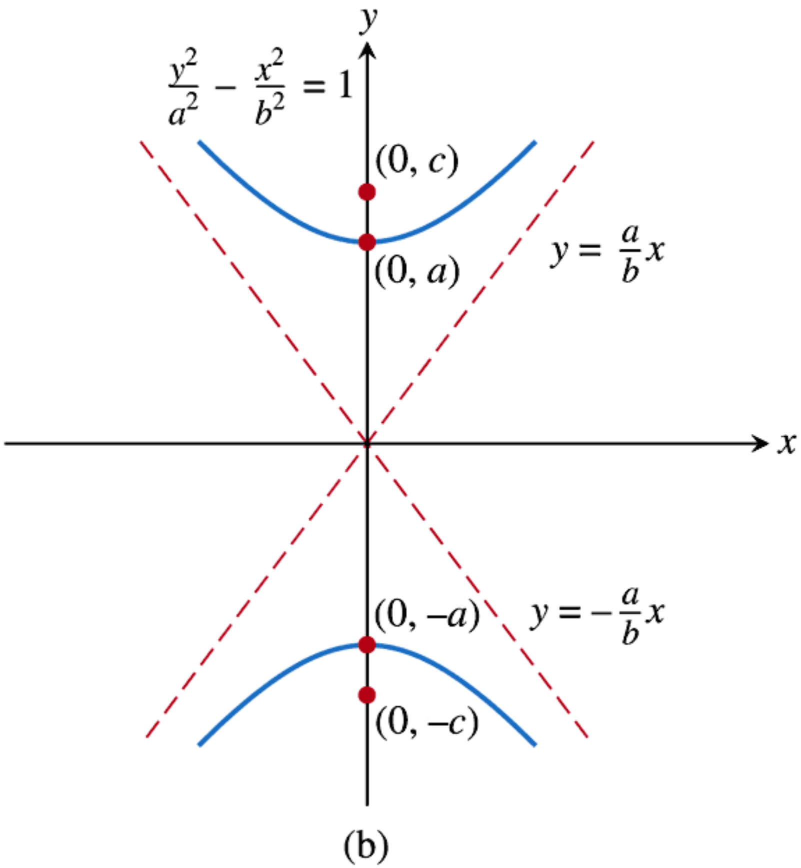


Hyperbola with Center (0,0)

• Standard equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
• Focal axis	x-axis	y-axis
• Foci	$(\pm c, 0)$	$(0, \pm c)$
• Vertices	$(\pm a, 0)$	$(0, \pm a)$
• Semitransverse axis	a	a
• Semiconjugate axis	b	b
• Pythagorean relation	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
• Asymptotes	$y = \pm \frac{b}{a} x$	$y = \pm \frac{a}{b} x$

Hyperbola Centered at (0,0)





Example **Finding the Vertices and Foci of a Hyperbola**

Find the vertices and the foci of the hyperbola

$9x^2 - 4y^2 = 36.$

Example **Finding the Vertices and Foci of a Hyperbola**

Find the vertices and the foci of the hyperbola

$9x^2 - 4y^2 = 36.$

Divide both sides of the equation by 36 to find the

standard form $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 36.$

So $a^2 = 4,\ b^2 = 9,$ and $c^2 = a^2 + b^2 = 13.$ Thus the

vertices are $(\pm 2, 0)$ and the foci are $(\pm\sqrt{13}, 0).$

Example **Finding an Equation of a Hyperbola**

Find an equation of the hyperbola with foci $(0,4)$ and $(0,-4)$ whose conjugate axis has length 2.

Example **Finding an Equation of a Hyperbola**

Find an equation of the hyperbola with foci (0,4) and $(0,-4)$ whose conjugate axis has length 2.

The center is at (0,0). The foci are on the y-axis with $c = 4$.

The semiconjugate axis is $b = 2/2 = 1$.

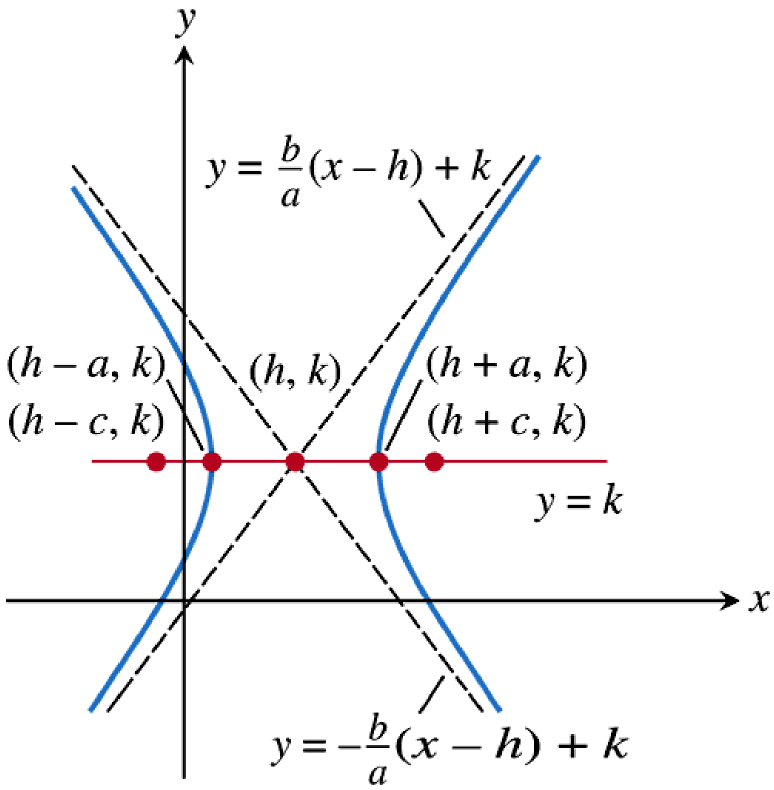
Thus $a^2 = c^2 - b^2 = 16 - 1 = 15$.

The standard form of the hyperbola is $\dfrac{y^2}{15} - \dfrac{x^2}{1} = 1$.

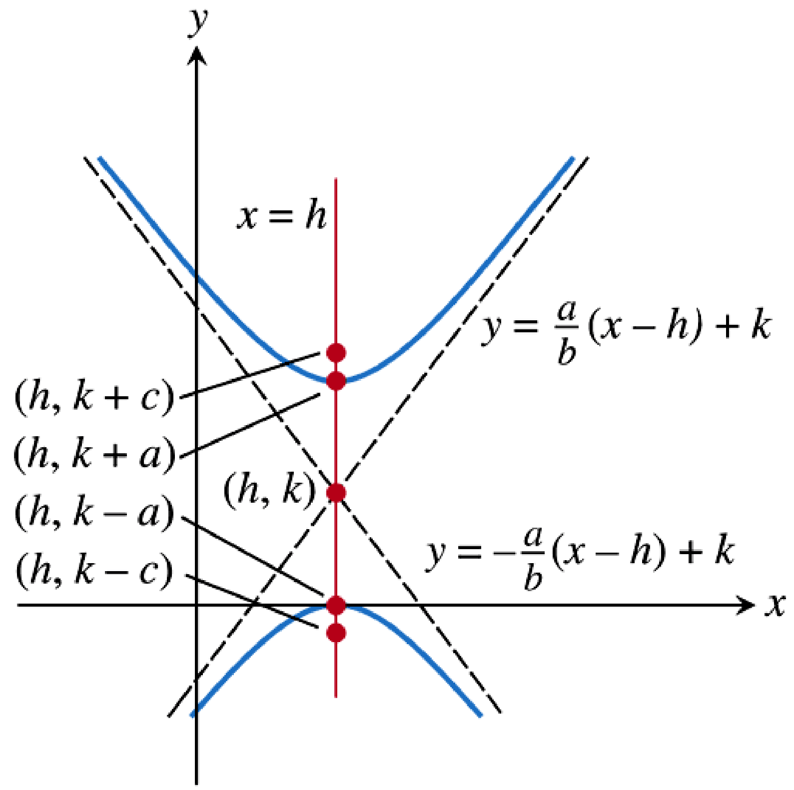
Hyperbola with Center (h,k)

• Standard equation	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
• Focal axis	$y=k$	$x=h$
• Foci	$(h\pm c,k)$	$(h,k\pm c)$
• Vertices	$(h\pm a,k)$	$(h,k\pm a)$
• Semimajor axis	a	a
• Semiminor axis	b	b
• Pythagorean relation	$c^2=a^2+b^2$	$c^2=a^2+b^2$
• Asymptotes	$y=\pm\frac{b}{a}(x-h)+k$	$y=\pm\frac{a}{b}(x-h)+k$

Hyperbola with Center (*h*,*k*)



(a)



(b)

Example **Finding an Equation of a Hyperbola**

Find the standard form of the equation for the hyperbola whose conjugate axis has endpoints (-1, 4) and (5, 4), and where the transverse axis has length 8.

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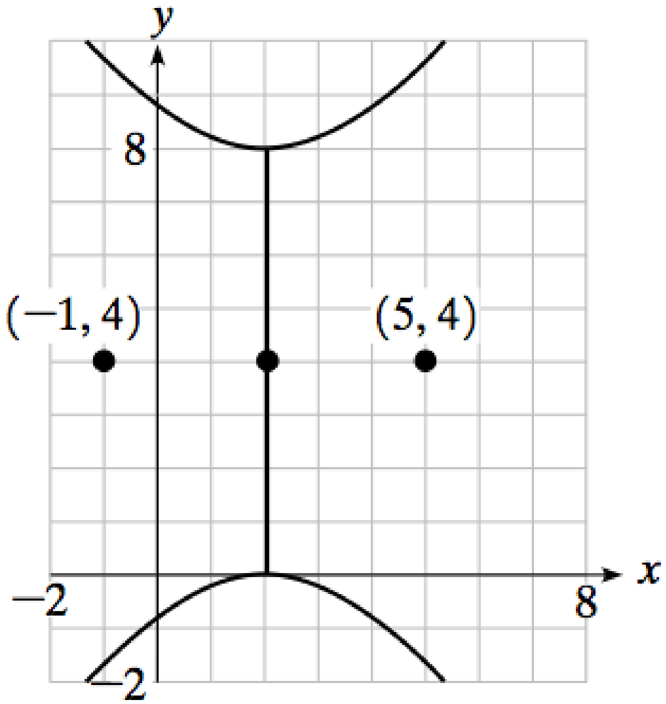
Find the standard form of the equation for the hyperbola whose conjugate axis has endpoints (-1, 4) and (5, 4), and where the transverse axis has length 8.

The figure shows the hyperbola

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Center is midpoint of minor axis, (2, 4).



Example Finding an Equation of a Hyperbola

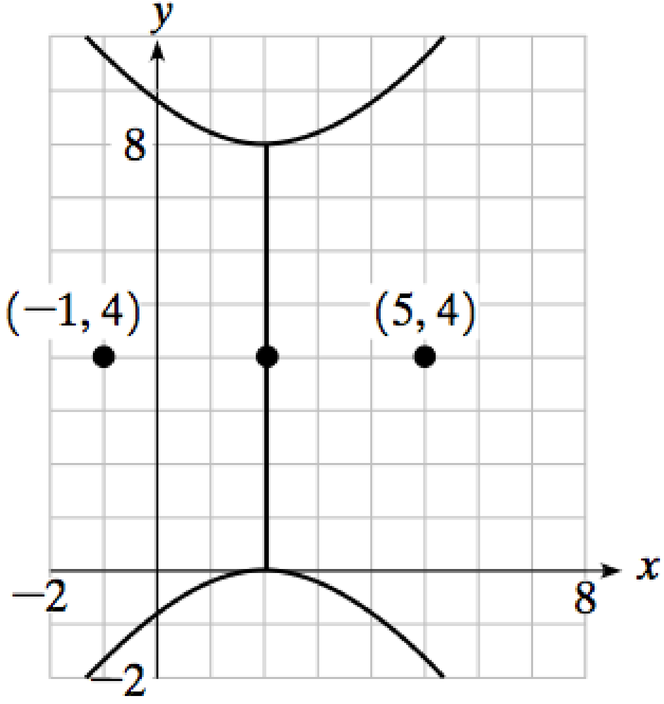
The semiminor axis and semimajor axis are

$$a = \frac{8}{2} = 4 \text{ and } b = \frac{5-(-1)}{2} = 3$$

The equation we seek is

$$\frac{(y-4)^2}{4^2} - \frac{(x-2)^2}{3^2} = 1$$

$$\frac{(y-4)^2}{16} - \frac{(x-2)^2}{9} = 1$$



Example **Locating Key Points of a Hyperbola**

Find the center, vertices, and foci of the hyperbola

$$\frac{(x+1)^2}{4} - \frac{y^2}{9} = 1.$$

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$$\frac{(x+1)^2}{4} - \frac{y^2}{9} = 1.$$

The center $(h,k) = (-1,0)$. Because the semitransverse axis $a = \sqrt{4} = 2,$ the vertices are at $(h \pm a, k) = \left(-1 \pm 2, 0\right)$ or $(-3,0)$ and $(1,0)$. Because $c = \sqrt{a^2 + b^2} = \sqrt{4+9} = \sqrt{13},$ the foci are at $(h \pm c, k) = (-1 \pm \sqrt{13}, 0)$ or approximately $(2.61,0)$ and $(-4.61,0)$.

Eccentricity of a Hyperbola

The **eccentricity** of a hyperbola is $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 + b^2}}{a}$,

where a is the semitransverse axis, b is the semiconjugate axis, and c is the distance from the center to either focus.

Quick Review

1. Find the distance between the points (a,b) and $(c,4)$.

2. Solve for y in terms of x. $\dfrac{y^2}{16}-\dfrac{x^2}{2}=1$

Solve for x algebraically.

3. $\sqrt{3x+12}-\sqrt{3x-8}=10$

4. $\sqrt{6x^2+12}-\sqrt{6x^2-1}=1$

5. Solve the system of equations:

$c-a=2$

$c^2-a^2=16a/c$

Quick Review Solutions

1. Find the distance between the points (a, b) and $(c, 4)$.

$\sqrt{(a-c)^2+(b-4)^2}$

2. Solve for y in terms of x. $\frac{y^2}{16}-\frac{x^2}{2}=1$ $y=\pm\sqrt{8x^2+16}$

Solve for x algebraically.

3. $\sqrt{3x+12}-\sqrt{3x-8}=10$ no solution

4. $\sqrt{6x^2+12}-\sqrt{6x^2-1}=1$ $x=\pm\frac{\sqrt{222}}{6}$

5. Solve the system of equations:

$c-a=2$

$c^2-a^2=16a/c$ no solution